

Viscosity spectral functions of the dilute Fermi gas in kinetic theory

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Abstract

We compute the viscosity spectral function of the dilute Fermi gas for different values of the s -wave scattering length a , including the unitarity limit $a \rightarrow \infty$. We perform the calculation in kinetic theory by studying the response to a non-trivial background metric. We find the expected structure consisting of a diffusive peak in the transverse shear channel and a sound peak in the longitudinal channel. At zero momentum the width of the diffusive peak is $\omega_0 \simeq (2\varepsilon)/(3\eta)$ where ε is the energy density and η is the shear viscosity. At finite momentum the spectral function approaches the collisionless limit and the width is of order $\omega_0 \sim k(T/m)^{1/2}$.

I. INTRODUCTION

The dilute Fermi gas at unitarity is a strongly correlated scale invariant quantum fluid [1, 2]. It provides a beautiful model system for testing ideas from many different areas of physics, including condensed matter physics, nuclear and particle physics, and quantum gravity. An important property of the Fermi gas at unitarity is nearly perfect fluidity [3]. Experiments with trapped Fermi gases in the vicinity of a Feshbach resonance indicate that [4–9]

$$\frac{\eta}{s} \lesssim 0.5 \hbar/k_B, \quad (1.1)$$

where η is the shear viscosity and s is the entropy density. This value is smaller than the viscosity to entropy density of any known fluid with the possible exception of the quark gluon plasma [10–12]. Equ. (1.1) approaches a bound $\eta/s = \hbar/(4\pi k_B)$ that has been proposed based on the AdS/CFT correspondence [13, 14].

The Kubo formula relates the shear viscosity to the zero momentum, zero frequency limit of the retarded stress tensor correlation function. In this work we will study the frequency and momentum dependence of this correlation function. We will focus on the regime of high temperature and small frequency and momentum where the methods of kinetic theory are applicable. Our study is motivated by several considerations. The first observation is that the frequency and momentum dependence of the retarded correlator encodes the time and distance scales over which the viscous contribution to the stress tensor relaxes to the Navier-Stokes expression. These scales are important for understanding how hydrodynamics breaks down in the dilute corona of a trapped Fermi gas [7, 15]. Kinetic theory predicts that in the dilute limit the shear viscosity is independent of density. This leads to a difficulty in the analysis of experiments involving scaling flows in which the velocity field is linear in the coordinates. The Navier-Stokes equation predicts that in this case the viscous stress tensor is a function of time, but constant in space. As a consequence the dissipative force, which is proportional to gradients of the stresses, vanishes and the dissipated heat, which is governed by the volume integral of the square of the stress tensor, is infinite. This problem can be resolved if a finite relaxation time for the dissipative stresses is taken into account. In this case the dissipative stress tensor vanishes in the dilute corona, the dissipated heat is finite, and there is a viscous force that counteracts the expansion of the system.

A second motivation is to understand the physics behind nearly perfect fluidity in the unitary Fermi, in particular the question whether the unitary gas can be described in terms of quasi-particles. In kinetic theory quasi-particles manifest themselves in terms of a peak in the spectral

function, but the quasi-particle peak may disappear in strong coupling. The spectral function is accessible through the behavior of the euclidean correlation function which can be computed in imaginary time Monte Carlo simulations. A calculation of the spectral function in kinetic theory serves as a useful default model for numerical calculations of the euclidean correlator.

This work extends and complements several recent studies of the transport properties of the dilute Fermi gas at unitarity. Calculations of transport coefficients in the context of kinetic theory can be found in [16–20], and a quasi-particle model is discussed in [21]. A diagrammatic calculation of the shear viscosity using the Kubo formula is described in [22]. This work also studies the spectral function. A sum rule for the spectral function of the stress tensor was proved in [23]. Related work on dilute Bose gases can be found in [24], and first attempts to extend the AdS/CFT correspondence to non-relativistic systems are described in [25, 26]. Finally, the spectral function of the stress tensor in QCD is studied in [27–31].

II. PRELIMINARIES

The retarded correlation function of the stress tensor is defined by

$$G_R^{ijkl}(\omega, \mathbf{k}) = -i \int dt \int d\mathbf{x} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \Theta(t) \langle [\Pi^{ij}(t, \mathbf{x}), \Pi^{kl}(0, \mathbf{0})] \rangle. \quad (2.1)$$

A careful definition of the stress tensor $\Pi^{\alpha\beta}(t, \mathbf{x})$ in terms of the Hamiltonian of the dilute Fermi gas was given in [22, 32]. In the following, we only need the definition of the stress tensor in terms of hydrodynamic and kinetic variables, see Sec. III and IV. The spectral function is defined by

$$\rho^{ijkl}(\omega, \mathbf{k}) = -2\text{Im} G^{ijkl}(\omega, \mathbf{k}). \quad (2.2)$$

In the following, we will focus on the longitudinal and transverse shear channels ρ^{xyxy} , ρ^{xxzz} and ρ^{zzzz} where the z -direction is the direction defined by the external momentum \mathbf{k} . We will compute the retarded correlation function using linear response theory. The external field that couples to the stress tensor is the metric $g_{ij}(t, \mathbf{x})$. A formalism for coupling an interacting theory of non-relativistic particles to an external metric in a way that exhibits a non-relativistic version of general coordinate invariance was recently developed in [33]. The response to a small perturbation $g_{ij}(t, \mathbf{x}) = \delta_{ij} + h_{ij}(t, \mathbf{x})$ around the flat metric is given by

$$\delta\Pi^{ij} = \frac{\delta\Pi_{ij}^{eq}}{\delta h_{ij}} h^{ij} - \frac{1}{2} G_R^{ijkl} h_{kl} \quad (2.3)$$

where Π_{ij}^{eq} is the expectation value of the stress tensor in equilibrium.

III. HYDRODYNAMICS

At very low frequency and momentum the retarded correlation function is governed by hydrodynamics. The hydrodynamic limit provides an important constraint for the kinetic description of the system. In this section we will derive these constraints in the case of correlations functions of the stress tensor. We note that hydrodynamics breaks down for frequencies $\omega \gtrsim (\eta/n)T$ and momenta $k \gtrsim (\eta/n)(mT)^{1/2}$, where n is the density and T is the temperature. In the unitary limit and in the vicinity of T_c we have $\eta/n \sim 1$ and the range of validity of hydrodynamics is very large. In the weak coupling limit $\eta \sim (mT)^{1/2}/a^2$, where a is the scattering length, and the range of validity of hydrodynamics is much smaller.

In order to study linear response theory we consider hydrodynamics in a curved background $g_{ij}(\mathbf{x}, t)$. Non-relativistic fluid dynamics in a non-trivial metric was studied in [34]. The continuity equation and the equations of momentum and energy conservation are given by

$$\begin{aligned} \frac{1}{\sqrt{g}} \partial_t (\sqrt{g} \rho) + \nabla_i (\rho v^i) &= 0, \\ \frac{1}{\sqrt{g}} \partial_t (\sqrt{g} \rho v^i) + \nabla_j \Pi^{ji} &= 0, \\ \frac{1}{\sqrt{g}} \partial_t (\sqrt{g} \rho s) + \nabla_i \left(\rho s v^i - \frac{\kappa}{T} \partial^i T \right) &= \frac{2R}{T}, \end{aligned} \quad (3.1)$$

where $\rho = mn$ is the mass density, v^i is the fluid velocity, and $g = \det(g_{ij})$. The covariant derivative of a vector field is given by $\nabla_i v_j = \partial_i v_j - \Gamma_{ij}^k v_k$, where Γ_{ij}^k is the Christoffel symbol associated with g_{ij} . The stress tensor is

$$\Pi^{ij} = \rho v^i v^j + P g^{ij} - \sigma^{ij}, \quad (3.2)$$

where the viscous term, σ_{ij} , is given by

$$\sigma_{ij} = \eta \left(\nabla_i v_j + \nabla_j v_i + \dot{g}_{ij} \right) + \left(\zeta - \frac{2}{3} \eta \right) g_{ij} \left(\nabla_k v^k + \frac{\dot{g}}{2g} \right). \quad (3.3)$$

The dissipative function R is given by

$$\begin{aligned} 2R &= \frac{\eta}{2} \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3} g_{ij} \nabla_k v^k + \dot{g}_{ij} - \frac{1}{3} g_{ij} \frac{\dot{g}}{g} \right)^2 \\ &\quad + \zeta \left(\nabla_i v^i + \frac{\dot{g}}{2g} \right)^2 + \frac{\kappa}{T} \partial_i T \partial^i T, \end{aligned} \quad (3.4)$$

where κ is the thermal conductivity. We are interested in small deviations from equilibrium. For this purpose we write $g_{ij} = \delta_{ij} + h_{ij}$, and $\rho = \rho_0 + \delta\rho$, $P = P_0 + \delta P$ etc. and linearize the equations

in h_{ij} , v_i , δP , δs , $\delta \rho$ and δT . After going to Fourier space we find

$$\begin{aligned}\frac{i}{2}\omega\rho_0 h + i\omega\delta\rho + \rho_0(-ik)v_z &= 0, \\ i\omega\rho_0 v^i + (-ik)\delta P\delta^{iz} + ik\sigma^{iz} &= 0, \\ i\omega\rho_0\delta s + \frac{\kappa k^2}{T}\delta T &= 0,\end{aligned}\tag{3.5}$$

where $h = \text{Tr}(h_{ij})$ and we have chosen the direction of k to be the z -axis. The fluctuations in thermodynamic variables are related by thermodynamic identities. We will consider $\delta\rho$ and δs to be the independent variable and use

$$\begin{aligned}\delta P &= \left.\frac{\partial P}{\partial\rho}\right|_s\delta\rho + \left.\frac{\partial P}{\partial s}\right|_\rho\delta s = c_s^2\delta\rho + \left.\frac{\partial P}{\partial s}\right|_\rho\delta s, \\ \delta T &= \left.\frac{\partial T}{\partial\rho}\right|_s\delta\rho + \left.\frac{\partial T}{\partial s}\right|_\rho\delta s = \left.\frac{\partial T}{\partial\rho}\right|_s\delta\rho + \frac{T}{c_V}\delta s,\end{aligned}\tag{3.6}$$

where c_s is the speed of sound and c_V is specific heat at constant volume.

We can now solve for the hydrodynamic variables in terms of the external field h_{ij} and compute Π_{ij} from Eq. (3.2). We then determine G_R using Eq. (2.3). We are particularly interested in the transverse shear correlators G_R^{xyxy} and G_R^{xzzx} , as well as the longitudinal correlator G_R^{zzzz} . We find

$$G_R^{xyxy} = -i\eta\omega,\tag{3.7}$$

$$G_R^{xzzx} = -\frac{i\eta\omega^2}{\omega - ik^2(\eta/\rho_0)},\tag{3.8}$$

$$G_R^{zzzz} = -\frac{\omega^2\rho_0(c_s^2 + i\omega\alpha)}{\omega^2 - c_s^2k^2 - i\omega k^2\alpha}.\tag{3.9}$$

We note that G_R^{xzzx} has a diffusive pole, where η/ρ_0 is the momentum diffusion coefficient, and G_R^{zzzz} has a sound pole where α is the sound attenuation coefficient,

$$\alpha\rho_0 = \frac{4}{3}\eta + \zeta + \frac{\kappa c_s^2 k^2}{\omega^2} \left(\frac{1}{c_V} - \frac{1}{c_P} \right).\tag{3.10}$$

Finally, we can take the imaginary parts and compute the spectral functions. In the Kubo limit $k = 0$, $\omega \rightarrow 0$ we find

$$\lim_{\omega, k \rightarrow 0} \frac{\rho_T}{2\omega} = \eta,\tag{3.11}$$

$$\lim_{\omega, k \rightarrow 0} \frac{\rho_L}{2\omega} = \frac{4}{3}\eta + \zeta,\tag{3.12}$$

where T, L labels the transverse and longitudinal components of the spectral functions, $\rho_T = \rho^{xyxy}, \rho^{xzzx}$ and $\rho_L = \rho^{zzzz}$.

IV. KINETIC THEORY

In this section we calculate the spectral function using kinetic theory. The basic object in kinetic theory is the distribution function $f(\mathbf{x}, \mathbf{p}, t)$ where $f = f_{\uparrow} = f_{\downarrow}$. The distribution function satisfies the Boltzmann equation. The kinetic equation in a curved background can be found by starting from the Boltzmann equation in general relativity [35, 36]

$$\frac{1}{p^0} \left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma_{\alpha\beta}^i p^\alpha p^\beta \frac{\partial}{\partial p^i} \right) f(t, \mathbf{x}, \mathbf{p}) = C[f], \quad (4.1)$$

where i, j, k are three-dimensional indices and μ, α, β are four-dimensional indices. In the non-relativistic limit $p^0 \simeq m$, $\Gamma_{00}^i \simeq 0$, and $\Gamma_{0j}^i \simeq \frac{1}{2} g^{ik} \dot{g}_{kj}$ [33]. We get

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - \left(g^{il} \dot{g}_{lj} p^j + \Gamma_{jk}^i \frac{p^j p^k}{m} \right) \frac{\partial}{\partial p^i} \right) f(t, \mathbf{x}, \mathbf{p}) = C[f], \quad (4.2)$$

We consider small deviations from equilibrium and write $f = f_0 + \delta f$ with $f_0(\mathbf{p}) = f_0(p^i p^j g_{ij} / (2mT))$. We also write $g_{ij} = \delta_{ij} + h_{ij}$ and linearize in h_{ij} and δf . We get

$$\left(\partial_t + \frac{p^i}{m} \partial_i \right) \delta f + \frac{f_0(1-f_0)}{2mT} p^i p^j \dot{g}_{ij} = C[\delta f]. \quad (4.3)$$

We will solve the linearized Boltzmann equation by making an ansatz for δf . We go to Fourier space and restrict ourselves to quadratic terms in p and write

$$\delta f(\omega, \mathbf{k}, \mathbf{p}) = \omega f_0(1-f_0) \frac{p^i p^j}{2mT} \frac{\xi_T h_{ij}^T + \xi_L h_{ij}^L}{\omega - \mathbf{v}_p \cdot \mathbf{k} + i\epsilon}, \quad (4.4)$$

where $\xi_{T,L} = \xi_{T,L}(\omega, \mathbf{k})$ and

$$h_{ij}^T = h_{ij} - \frac{1}{3} \delta_{ij} h, \quad h_{ij}^L = \frac{1}{3} \delta_{ij} h. \quad (4.5)$$

Using this ansatz the LHS of the linearized Boltzmann equation becomes

$$LHS = \frac{i\omega f_0(1-f_0)}{2mT} \{ (\xi_T + 1) p^i p^j h_{ij}^T + (\xi_L + 1) p^i p^j h_{ij}^L \}. \quad (4.6)$$

The RHS of the Boltzmann equation involves the linearized collision integral

$$RHS = \int d\Gamma_{234} r(1, 2; 3, 4) \frac{\omega}{2mT} [g_{ij}(\mathbf{p}_1) + g^{ij}(\mathbf{p}_2) - g^{ij}(\mathbf{p}_3) - g^{ij}(\mathbf{p}_4)] (\xi_T h_{ij}^T + \xi_L h_{ij}^L), \quad (4.7)$$

where we have defined $g^{ij}(\mathbf{q}) \equiv g^{ij}(\mathbf{q}; \omega, \mathbf{k}) = q^i q^j / (\omega - \mathbf{v}_q \cdot \mathbf{k} + i\epsilon)$ and we have labeled the momenta such that $\mathbf{p}_1 = \mathbf{p}$. We have also defined the phase space measure $d\Gamma_{234} = d\Gamma_1 d\Gamma_2 d\Gamma_3$ with $d\Gamma_i = d^3 p_i / (2\pi)^3$ and the transition rate $r(1, 2; 3, 4) = w(1, 2; 3, 4) D(1, 2; 3, 4)$. The factor $D(1, 2; 3, 4)$ contains the distribution functions

$$D(1, 2; 3, 4) = f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) (1 - f_0(\mathbf{p}_3)) (1 - f_0(\mathbf{p}_4)), \quad (4.8)$$

and $w(1, 2; 3, 4)$ is the collision probability

$$w(1, 2; 3, 4)d\Gamma_{34} = v_{rel} \left(\frac{d\sigma}{d\Omega'} \right) d\Omega'. \quad (4.9)$$

Here, $v_{rel} = |\mathbf{v}_1 - \mathbf{v}_2|$ is the relative velocity, $d\Omega'$ is the solid angle along $2\mathbf{q}' = \mathbf{p}_3 - \mathbf{p}_4$ and $(d\sigma)/(d\Omega')$ is the differential cross section. We also define $2\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$ where $|\mathbf{q}| = |\mathbf{q}'|$. We will consider a dilute Fermi gas in which the scattering amplitude is completely characterized by the s -wave scattering length a . The differential cross section is

$$\frac{d\sigma}{d\Omega'} = \frac{1}{4\pi} \frac{a^2}{1 + a^2 q^2}. \quad (4.10)$$

In order to solve for $\xi_{T,L}$ we will take moments of the linearized Boltzmann equation. The quadratic moments of the LHS are given by

$$L^{ab} = \int d\Gamma_p \frac{i\omega f_0(1 - f_0)}{2mT} p^a p^b p^i p^j \sum_{T,L} h_{ij}^{T,L} (\xi_{T,L} + 1), \quad (4.11)$$

where $L^{ab} = L^{ab}(\omega, \mathbf{k})$. The moments of the RHS are given by

$$\begin{aligned} R^{ab} &= \frac{\omega}{2mT} \int d\Gamma_{tot} r(1, 2; 3, 4) p_1^a p_1^b [g_{ij}(\mathbf{p}_1) + g^{ij}(\mathbf{p}_2) - g^{ij}(\mathbf{p}_3) - g^{ij}(\mathbf{p}_4)] \sum_{T,L} \xi_{T,L} h_{ij}^{T,L} \\ &= \frac{\omega}{2mT} \int d\Gamma_{tot} r(1, 2; 3, 4) [p_1^a p_1^b + p_2^a p_2^b - p_3^a p_3^b - p_4^a p_4^b] g^{ij}(\mathbf{p}_1) \sum_{T,L} \xi_{T,L} h_{ij}^{T,L} \\ &= \frac{2\omega}{m^2 T} \int \frac{d^3 p d^3 q}{(2\pi)^6} d\Omega_{q'} \frac{a^2 q}{1 + a^2 q^2} D_{2\leftrightarrow 2} (q^a q^b - q'^a q'^b) g^{ij}(\mathbf{p}_1) \sum_{T,L} \xi_{T,L} h_{ij}^{T,L}, \end{aligned} \quad (4.12)$$

with $D_{2\leftrightarrow 2} = D(1, 2; 3, 4)$ and $R^{ab} = R^{ab}(\omega, \mathbf{k})$.

A. Transverse channel

We can now determine $\xi_{T,L}$ by solving $L^{ab} = R^{ab}$. We begin by considering $h_{xy} = h_{yx} \neq 0$ with all other components of h_{ij} vanishing. In this case $h = 0$ and only ξ_T contributes. We find

$$\begin{aligned} L^{xy} &= \int d\Gamma_p \frac{i\omega f_0(1 - f_0)}{2mT} (\xi_T + 1) p^x p^y p^i p^j h_{ij}^T \\ &= \frac{i\omega (\xi_T + 1) h_{xy}}{30\pi^2 mT} \int dp p^6 f_0(1 - f_0) \equiv \frac{i\omega (\xi_T + 1) h_{xy}}{15} I_{xy} \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} R^{xy} &= \frac{2\omega \xi_T h_{xy}}{m^2 T} \int \frac{d^3 p d^3 q}{(2\pi)^6} d\Omega_{q'} \frac{a^2 q}{1 + a^2 q^2} D_{2\leftrightarrow 2} (q^x q^y - q'^x q'^y) g^{ij}(\mathbf{p}_1) \\ &= \frac{2\omega \xi_T h_{xy}}{\pi^3 mT} \int dp dq d\alpha d\beta \frac{a^2 q}{1 + a^2 q^2} D_{2\leftrightarrow 2} \frac{p^2 q^2 q^x q^y q^x q^y}{2m\omega - pk\alpha - 2qk\beta + i\epsilon} \equiv \frac{\omega \xi_T h_{xy}}{15} C_{xy}. \end{aligned} \quad (4.14)$$

Setting $L^{xy} = R^{xy}$ gives

$$\xi_T^{(xy)}(\omega, \mathbf{k}) = -\frac{I_{xy}}{I_{xy} + iC_{xy}(\omega, \mathbf{k})}. \quad (4.15)$$

The superscript (xy) indicates that we have obtained ξ_T by projecting on the Π^{xy} channel. In the limit $\mathbf{k} \rightarrow 0$ our ansatz Eq. (4.4) is complete, and there is no dependence on the channel. At finite \mathbf{k} , however, additional tensor structures in Eq. (4.4) are possible and there is some dependence of $\xi_{L,T}$ on the channel.

In kinetic theory the stress tensor is given by

$$\delta\Pi^{ij} = 2 \int \frac{d^3p}{(2\pi)^3} \frac{p^i p^j}{m} \delta f, \quad (4.16)$$

where the factor 2 is the spin degeneracy. Using the ansatz Eq. (4.4) we get

$$\delta\Pi^{xy} = \frac{\omega \xi_T^{(xy)} h_{xy}}{mT} \int \frac{d^3p}{(2\pi)^3} f_0(1-f_0) \frac{p^x p^y p^x p^y}{m\omega - p^z k + i\epsilon} \equiv \xi_T^{(xy)} h_{xy} J_{xy}. \quad (4.17)$$

We can now extract the retarded correlation function and the spectral function using Eq. (2.3) and Eq. (2.2). We find

$$\rho^{xyxy}(\omega, \mathbf{k}) = 2\text{Im}(\xi_T^{(xy)}(\omega, \mathbf{k}))\text{Re}(J_{xy}(\omega, \mathbf{k})) + 2\text{Re}(\xi_T^{(xy)}(\omega, \mathbf{k}))\text{Im}(J_{xy}(\omega, \mathbf{k})). \quad (4.18)$$

In general the functions C_{xy} and J_{xy} have to be computed numerically. The limit $\mathbf{k} \rightarrow 0$ can be studied analytically. We may consider two cases. In the free case the collision integral C_{xy} vanishes and $\xi_T = -1$. The spectral function is then given by $\rho^{xyxy}(\omega, \mathbf{k}) = -2\text{Im} J_{xy}(\omega, \mathbf{k})$. The function $J_{xy}(\omega, \mathbf{k})$ can be computed analytically for all values of ω and \mathbf{k} . The collisionless spectral function is

$$\frac{\rho^{xyxy}(\omega, \mathbf{k})}{2\omega} = \frac{1}{16\pi} \frac{(2mT)^2}{k} Li_2\left(\zeta_F^{-1} \exp\left(-\frac{m\omega^2}{2Tk^2}\right)\right), \quad (4.19)$$

where $Li_2(x)$ is a polylogarithm and $\zeta_F = \exp(-\mu/T)$ is the fugacity. In the limit $\mathbf{k} \rightarrow 0$ we find

$$\lim_{\mathbf{k} \rightarrow 0} \frac{\rho^{xyxy}(\omega, \mathbf{k})}{2\omega} = \frac{\varepsilon}{3} \pi \delta(\omega), \quad (4.20)$$

where ε is the energy density

$$\varepsilon = 2 \int \frac{d^3p}{(2\pi)^3} f_0 \frac{p^2}{2m}. \quad (4.21)$$

At finite momentum the delta function is spread out over the regime $\omega \lesssim |\mathbf{k}|(T/m)^{1/2}$. In the collisional case we find that in the limit $\mathbf{k} \rightarrow 0$ the collision integral is real and scales as $C_{xy}(\omega, \mathbf{k} \rightarrow$

0) $\sim 1/\omega$. This means that the spectral function is dominated by the $\text{Im}(\xi_T^{(xy)}(\omega, \mathbf{k}))\text{Re}(J_{xy}(\omega, \mathbf{k}))$ term. We also find that in this limit $\text{Re}(J_{xy}) \simeq I_{xy}/(15m)$. This implies that

$$\eta(\omega) \equiv \lim_{\mathbf{k} \rightarrow 0} \frac{\rho^{xyxy}(\omega, \mathbf{k})}{2\omega} = \frac{I_{xy}^2}{15m\omega} \frac{\text{Re}(C_{xy}(\omega, \mathbf{0}))}{I_{xy}^2 + \text{Re}(C_{xy}(\omega, \mathbf{0}))^2}. \quad (4.22)$$

The zero frequency limit is

$$\eta \equiv \lim_{\omega \rightarrow 0} \eta(\omega) = \frac{I_{xy}^2}{15m\omega \text{Re}(C_{xy}(\omega, \mathbf{0}))} = \frac{15(mT)^{3/2}}{32\sqrt{\pi}} \begin{cases} 1 & a \rightarrow \infty \\ 1/(3mTa^2) & a \rightarrow 0 \end{cases}, \quad (4.23)$$

where in the last step we have evaluated the integrals in the high temperature limit. This result agrees with the hydrodynamic limit given in Eq. (3.7) and the known formula for the shear viscosity of a dilute Fermi gas in kinetic theory [16]. We can now write the frequency dependence as

$$\eta(\omega) = \frac{\eta}{1 + \omega^2\tau^2} \quad (4.24)$$

where $\tau = I_{xy}/(\omega \text{Re}(C_{xy})) = 15m\eta/I_{xy}$ is a relaxation time. In the high temperature limit τ agrees with the result obtained in [15]. We can also evaluate the sum rule

$$\frac{1}{\pi} \int_0^\infty d\omega \eta(\omega) = \frac{\eta}{2\tau} = \frac{I_{xy}}{30m} = \frac{\varepsilon}{3}. \quad (4.25)$$

We note that the sum rule agrees with the integral of the free spectral function given in Eq. (4.20). The sum rule also agrees with the general sum rule obtained in [23] if we take account the fact that kinetic theory does not reproduce the short distance contribution $\eta(\omega) \sim C/\sqrt{m\omega}$, where C is Tan's contact [37].

The spectral function in the xz channel can be computed in the same fashion. The spectral function $\rho^{xzxz}(\omega, \mathbf{k})$ is given by Eq. (4.18) with the obvious replacement $(xy) \rightarrow (xz)$. In the Kubo limit we find

$$\lim_{\omega, \mathbf{k} \rightarrow 0} \frac{\rho^{xzxz}(\omega, \mathbf{k})}{2\omega} = \eta \quad (4.26)$$

as expected from hydrodynamics, see Eq. (3.8). Numerical results for the spectral function in the xy and xz channel are shown in Fig. 1. We consider two different scattering length, one corresponding to weak coupling, $k_F a = 0.1$, and one close to unitarity, $k_F a = 100$. We plot the spectral function as a function of ω/T for several values of $\bar{k} = k/\sqrt{2mT}$ and T/T_F .

We first discuss $\rho^{xyxy}(\omega, \mathbf{k})$. At zero momentum the spectral function normalized to $2\eta\omega$ is a Lorentzian with a width proportional to $1/\eta$. This implies that the peak in the spectral function

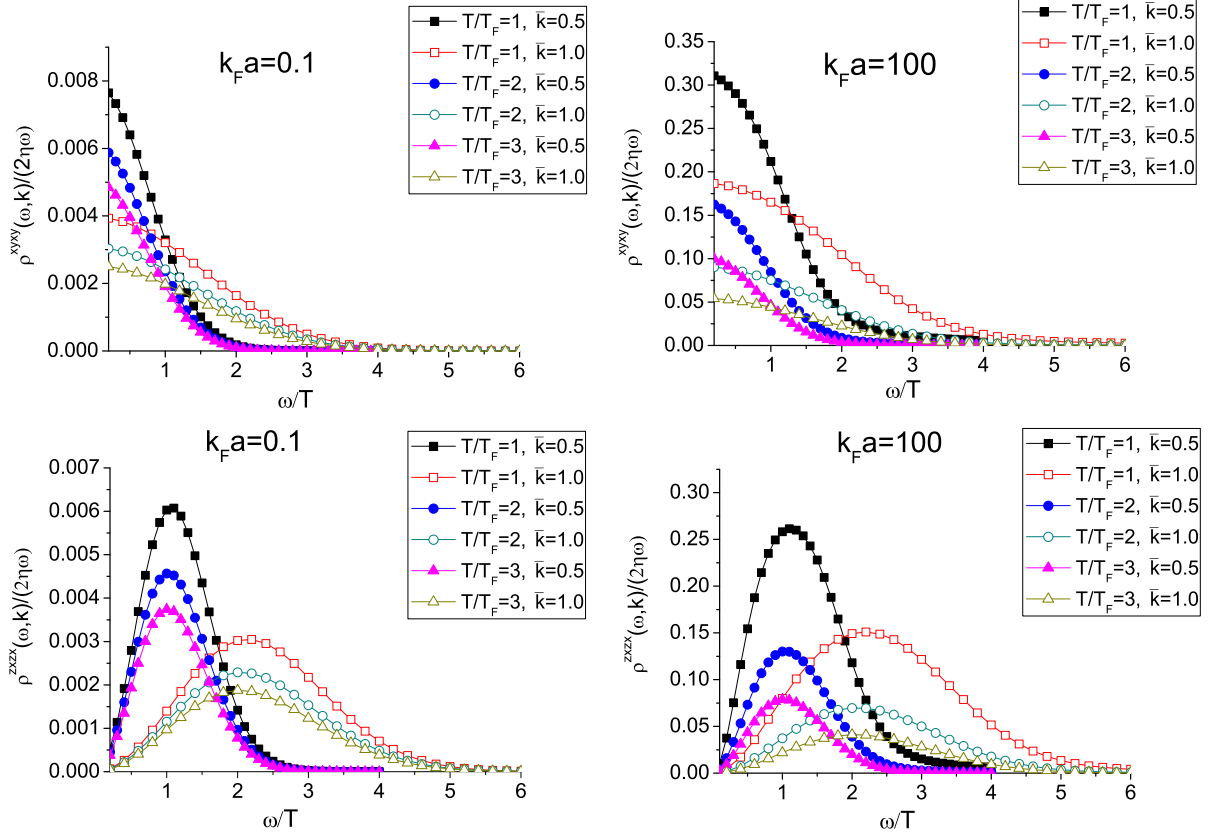


FIG. 1: Stress tensor spectral functions in the xy and xz channels. We plot the spectral function $\rho(\omega, k)$ normalized to the value in the Kubo limit, $\rho(\omega \rightarrow 0, 0) = 2\eta\omega$, as a function of ω/T for different values of $\bar{k} = k/(2mT)^{1/2}$ and T/T_F . The panels on the left show the result for $k_F a = 0.1$, which corresponds to weak coupling, and the panels on the right show the spectral function close to unitarity, for $k_F a = 100$.

is narrow in weak coupling, but very broad in strong coupling. As the momentum k grows the width of the peak increases and it becomes less sensitive on the strength of the coupling. In this regime the spectral function is close to the free spectral function. We also find numerically that the spectral function satisfies the sum rule Eq. (4.25) even at non-zero momentum k . In the xz channel the normalized spectral function $\rho^{xxx}(\omega, \mathbf{k})/(2\eta\omega)$ vanishes in the limit $\omega \rightarrow 0$ for any non-zero value of k . For $k = 0$ the normalized spectral function approaches unity as $\omega \rightarrow 0$. This is in agreement with the hydrodynamic prediction Eq. (3.8). We find that the spectral function has a peak which, like the peak in the xy channel, corresponds to a diffusive mode. The width of the mode is controlled by k except in the limit of small momenta where the width is controlled by the shear viscosity.

B. Longitudinal case

In this section we study the longitudinal spectral function $\rho^{zzzz}(\omega, \mathbf{k})$. The calculation is more involved because both ξ_T and ξ_L contribute. We consider a perturbation $h_{zz} \neq 0$ with all other $h_{ij} = 0$. In this case $h_{ij}^L = \frac{h_{zz}}{3}\delta_{ij}$ and $h_{zz}^T = h_{ij} - \frac{h_{zz}}{3}\delta_{ij} = -\frac{h_{zz}}{3}\text{diag}(1, 1, -2)$. The LHS of the Boltzmann equations is $L^{zz} = L_T^{zz} + L_L^{zz}$ with

$$L_{T,L}^{zz} = \int d\Gamma \frac{i\omega f_0(1-f_0)}{2mT} (\xi_{T,L} + 1) p^z p^z p^i p^j h_{ij}^{T,L} \equiv \frac{i\omega(\xi_{T,L} + 1)h_{zz}}{3} I_{T,L}^{zz}. \quad (4.27)$$

The collision integrals can be written as

$$\begin{aligned} R_T^{zz} &= \frac{2\omega\xi_T}{m^2T} \int \frac{d^3p d^3q}{(2\pi)^6} d\Omega_{q'} D_{2\leftrightarrow 2} \frac{a^2q}{1+a^2q^2} (q^z q^z - q'^z q'^z) g^{ij}(\mathbf{p}_1) h_{ij}^T \\ &= -\frac{\omega\xi_T h_{zz}}{12\pi^3 mT} \int dp dq d\alpha d\beta D_{2\leftrightarrow 2} \frac{a^2q}{1+a^2q^2} p^2 q^4 \frac{(\beta^2 - \frac{1}{3})(p^2(1-3\alpha^2) - 8pq\alpha\beta + 4q^2(1-3\beta^2))}{2m\omega - pk\alpha - 2qk\beta + i\epsilon} \\ &\equiv \frac{\omega\xi_T h_{zz}}{3} C_T^{zz}, \end{aligned} \quad (4.28)$$

$$\begin{aligned} R_L^{zz} &= \frac{2\omega\xi_L}{m^2T} \int \frac{d^3p d^3q}{(2\pi)^6} d\Omega_{q'} D_{2\leftrightarrow 2} \frac{a^2q}{1+a^2q^2} (q^z q^z - q'^z q'^z) g^{ij}(\mathbf{p}_1) h_{ij}^L \\ &= -\frac{\omega\xi_L h_{zz}}{3\pi^3 mT} \int dp dq d\alpha d\beta D_{2\leftrightarrow 2} \frac{a^2q}{1+a^2q^2} p^2 q^4 \frac{(\beta^2 - \frac{1}{3})(p^2/4 + pq\alpha\beta + q^2)}{2m\omega - pk\alpha - 2qk\beta + i\epsilon} \\ &\equiv \frac{\omega\xi_L h_{zz}}{3} C_L^{zz}. \end{aligned} \quad (4.29)$$

We can now solve for $\xi_{T,L}$. We find

$$\xi_{T,L}^{(zz)}(\omega, \mathbf{k}) = -\frac{I_{T,L}^{zz}}{I_{T,L}^{zz} + iC_{T,L}^{zz}(\omega, \mathbf{k})}. \quad (4.30)$$

The zz component of the stress tensor is given by

$$\begin{aligned} \delta\Pi^{zz} &= 2\omega \int \frac{d^3p}{(2\pi)^3} f_0(1-f_0) \frac{p^z p^z}{m} \frac{p^i p^j}{2mT} \frac{\xi_T^{(zz)} h_{ij}^T + \xi_L^{(zz)} h_{ij}^L}{\omega - \mathbf{v}_p \cdot \mathbf{k} + i\epsilon} \\ &= \frac{\omega h_{zz}}{12\pi^2 mT} \int dp d\alpha \frac{f_0(1-f_0)p^6\alpha^2}{m\omega - pk\alpha + i\epsilon} \left[\xi_T^{(zz)}(3\alpha^2 - 1) + \xi_L^{(zz)} \right] \\ &\equiv \omega h_{zz} \left(\xi_T^{(zz)}(\omega, \mathbf{k}) J_T^{zz}(\omega, \mathbf{k}) + \xi_L^{(zz)}(\omega, \mathbf{k}) J_L^{zz}(\omega, \mathbf{k}) \right). \end{aligned} \quad (4.31)$$

We can now extract the retarded correlator and the spectral function using Eq. (2.3) and Eq. (2.2).

We get

$$\frac{\rho^{zzzz}(\omega, \mathbf{k})}{2\omega} = \text{Im} \left(\xi_T^{(zz)}(\omega, \mathbf{k}) J_T^{zz}(\omega, \mathbf{k}) + \xi_L^{(zz)}(\omega, \mathbf{k}) J_L^{zz}(\omega, \mathbf{k}) \right). \quad (4.32)$$

The result again simplifies in the limit $\mathbf{k} \rightarrow 0$. In the collisional case we find that the function J is

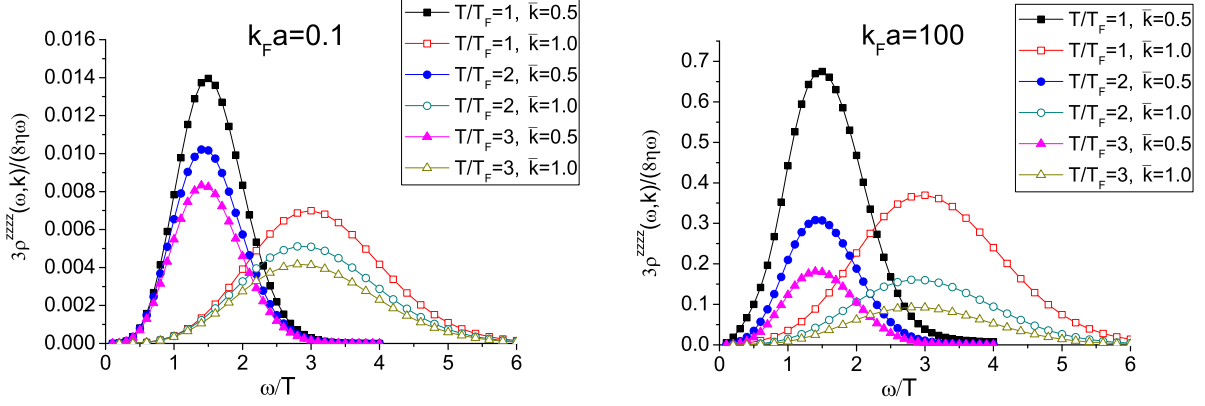


FIG. 2: Viscosity spectral function in the zz channel. We plot the spectral function $\rho(\omega, k)$ normalized to the value in the Kubo limit, $\rho(\omega \rightarrow 0, 0) = \frac{8}{3}\eta\omega$, as a function of ω/T for different values of $\bar{k} = k/(2mT)^{1/2}$ and T/T_F .

dominated by its real part. Also, one can show that the $\mathbf{k} \rightarrow 0$ limit of $\text{Re}(C_L^{zz}) = 0$ and therefore $\text{Im}(\xi_L^{(zz)}) = 0$. We find

$$\lim_{\mathbf{k} \rightarrow 0} \frac{\rho^{zzzz}(\omega, \mathbf{k})}{2\omega} = \lim_{\mathbf{k} \rightarrow 0} \text{Im}(\xi_T^{(zz)}(\omega, \mathbf{k})) \text{Re}(J_T^{zz}(\omega, \mathbf{k})) = \frac{I_T^{zz} \text{Re}(J_T^{zz}(\omega, \mathbf{0}))}{(I_T^{zz})^2 + \text{Re}(C_T^{zz}(\omega, \mathbf{0}))^2} \quad (4.33)$$

The transverse and longitudinal integrals that we have done are obviously related to each other. In the limit $\mathbf{k} \rightarrow 0$ we have

$$I_T^{zz} = \frac{2}{15} I_{xy} \quad J_T^{zz}(\omega, \mathbf{0}) = \frac{4}{3} J_{xy}(\omega, \mathbf{0}) = \frac{4}{45m\omega} I_{xy} \quad C_T^{zz} = \frac{2}{15} C_{xy}. \quad (4.34)$$

This shows that as $\mathbf{k} \rightarrow 0$ the ansatz Eq. (4.4) is complete and $\xi_T^{(zz)}(\omega) = \xi_T^{(xy)}(\omega)$. We can also show that

$$\lim_{\omega, \mathbf{k} \rightarrow 0} \frac{\rho^{zzzz}(\omega, \mathbf{k})}{2\omega} = \frac{4}{3} \eta. \quad (4.35)$$

Comparing this result with the hydrodynamic prediction (3.9) we find that the bulk viscosity ζ vanishes. This follows from scale invariance at unitarity [34]. The fact that we also find $\zeta = 0$ away from unitarity is related to the fact that we only take into account elastic $2 \leftrightarrow 2$ collisions. Numerical results for $\rho^{zzzz}(\omega, \mathbf{k})$ normalized to the hydrodynamic limit $\frac{8}{3}\omega\eta$ are shown in Fig. 2. We observe that the spectral function shows a sound peak at $\omega = c_s k$. At the largest temperature considered, $T/T_F = 3$, the speed of sound differs from the asymptotic high temperature value $c_s^2 = (5T)/(3m)$ by about 10%. The width of the sound peak is close to that of the free spectral function. This is at variance with hydrodynamics which predicts that the sound peak is narrow in

strong coupling, and in the limit of small momentum. At zero momentum hydrodynamics predicts that $\rho(\omega)/\omega$ is constant. Higher order effects will turn this part of the spectrum into a broad peak with a width inversely proportional to the relaxation time. This implies that at small momentum we expect a narrow peak with a width proportional to $(k/c_s)(\eta/\rho)$ superimposed on a broad peak with a width proportional $\varepsilon/(T\eta)$. Our ansatz correctly reproduces the $k \rightarrow 0$ limit as well as the existence of a sound peak, but it appears to be too simple to reproduce the two-peak structure at small k .

V. SUMMARY AND DISCUSSION

We have computed the viscosity spectral function using kinetic theory in a non-trivial background metric. We find that a simple quadratic ansatz for the non-equilibrium distribution function δf satisfies the main constraints on the spectral function, in particular the existence of a diffusive peak and a sound peak (although it misses the interplay between these two features in the longitudinal channel), the correct Kubo limit, and the viscosity sum rule. Kinetic theory misses the non-analytic high frequency tail of the spectral function $\eta(\omega) \sim C/\sqrt{m\omega}$. We find that the width of the diffusive peak at zero momentum is inversely proportional to the viscosity. This implies that the relaxation time for the shear stress is $\tau \simeq 3\eta/(2\varepsilon)$, in agreement with [15]. We also find that at large momentum the spectral function is close to the free spectral function, and the width of the transport peak is of order $k(T/m)^{1/2}$.

There are at least two important questions that we have not addressed in this paper. The first is a systematic matching to second order hydrodynamics. For this purpose it is important to understand the structure of second order hydrodynamics in the conformal limit along the lines of the analysis in the relativistic case performed in [38]. The second is a detailed study of the imaginary time correlation functions, with an emphasis on finding the optimal strategy for extracting transport coefficients from Quantum Monte Carlo calculations. The main difficulty is that the imaginary time correlation function is mainly sensitive to the spectral weight of the transport peak, and that the sum rule implies that the weight is independent of the shear viscosity. This problem can be addressed by utilizing the information contained in the momentum dependence of the correlation function, and by considering not only the shear channel but also the sound channel [27, 30].

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